3

Dimensional Analysis

1.0 INTRODUCTION

Many engineering problems are too complex to solve in closed mathematical form. In such cases, a type of analysis, which involves the dimensions of the quantities entering the problem, may be useful. This is called dimensional analysis. Uses for dimensional analysis include the following:

- To reduce the number of variables to be studied or plotted
- In planning experiments
- In designing engineering models to be built and studied and in interpreting model data
- To emphasize the relative importance of parameters entering a problem
- To enable units of measurement to be changed from one system to another

The last of these is a common, although relatively trivial, application.

Before discussing the details of dimensional analysis, it is useful to consider the nature of the end result. This will be done in terms of the projectile problem considered in the previous chapter (Fig. 2.10) where it was found that the elevation of the projectile would be as follows [see Eq. (2.24)]:

$$y = -gt^2/2 + v_0 t \sin \theta$$

where t is the time elapsed after launching the projectile with an initial velocity, v_0, inclined at an angle (θ) to the horizontal.

In considering this problem by the method of dimensions, it would first be observed that the main dependent variable (y) will be some function (ψ_1) of v_0, t, and g, i.e.,

Eq. (3.1) $$y = \psi_1(v_o, t, g)$$

After performing the dimensional analysis, the following result would be obtained using the procedure outlined later in this chapter:

Eq. (3.2) $$y/gt^2 = \psi_2(v_0/gt)$$

where ψ_2 is some function of the quantity in parenthesis. The number of variables has been reduced from four (y,v_0, t, and g) to two (y/gt^2 and v_0/gt), but ψ_2 must still be evaluated either experimentally or by further analysis.

Dimensional analysis thus represents a partial solution to the problem but one that is very useful and easily obtained. In the solution of complex problems, it can play a valuable role.

2.0 DEFINITIONS

Before considering the method of proceeding from Eq.(2.24) to Eq. (3.2) in this chapter, a few quantities will be defined. Fundamental or primary dimensions are properties of a system under study that may be considered independent of the other properties of interest. For example, there is one fundamental dimension in any geometry problem and this is length (L). The fundamental dimensions involved in different classes of mechanical problems are listed in Table 3.1 where L stands for length, F for force, and T for time. Dimensions other than F, L, and T, which are considered fundamental in areas other than mechanics, include: temperature

(θ), heat (H), electrical charge (Q), magnetic pole strength (P), chemical yield (Y), and unit cost (¢).

Table 3.1. Fundamental Dimensions for Different Types of Problems

Type of Problem	**Fundamental Dimensions**
Geometry	L
Statics	F
*Temporal	T
Kinematics	L, T
Work or Energy	F, L
Momentum or Impulse	F, T
Dynamics	F, L, T

*Temporal problems are those involved in time study, frequency analysis, time tables, etc.

Dimensional equations relate the dimensions of the fundamental quantities entering a problem to nonfundamental or secondary quantities. For example, acceleration (a) is a quantity of importance in most kinematic problems and the dimensions of acceleration are related to fundamental dimensions L and T by the following dimensional equation:

$$a = [LT^{-2}]$$

It is customary to enclose the combination of fundamental dimensions in brackets when writing a dimensional equation.

Dimensional units are the basic magnitudes used to specify the size of a fundamental quantity. In engineering problems, lengths are frequently measured in feet and times in seconds. In this system of dimensional units, acceleration would be expressed in units of ft/sec^2.

A nondimensional quantity is one whose dimensional equation has unity or $F^0L^0T^0$ on the right side. The quantities (y/gt^2) and (v_0/gt) in Eq. (3.2) are nondimensional. A group of variables that cannot be combined to form a nondimensional group is said to be dimensionally independent.

3.0 FUNDAMENTAL QUANTITIES

The choice of fundamental quantities is somewhat arbitrary, as the following discussion will reveal. In Table 3.1, the fundamental quantities for the general dynamics problem were stated as F, L, and T. The quantities M (mass), L, and T could also have been used. Then F would be a secondary variable related to fundamental variables M, L, and T by a dimensional equation based on Newton's second law:

$$F = ma = [MLT^{-2}]$$

F, L, and T will be used here since most engineering quantities are expressed in units of force and not in units of mass.

The principle of dimensional homogeneity first expressed by Fourier (1822) states that the dimensions of each term of any physically correct equation must be the same. For example, Eq. (2.24) is readily found to meet this test:

$$[L] = [(L/T^2)(T^2)] - [(L/T)(T)]$$

4.0 PROCEDURE

We are now ready to return to the projectile problem and consider the steps involved between Eq. (2.24) and Eq. (3.2). Since this is a kinematics problem there are two fundamental dimensions (L, T). Table 3.2 gives the fundamental dimensions for all quantities entering the problem.

The quantities g and t are dimensionally independent (may not be combined by raising to powers and multiplying together to form a nondimensional group). If a function such as Eq. (3.2) exists, then the following may also be written:

Eq. (3.3) $$y/gt^2 = \psi_2(g, t, v_0/gt)$$

The left side of this equation is nondimensional, and by the principle of dimensional homogeneity, all terms of ψ_2 must also be nondimensional.

The quantities g and t, therefore, cannot appear in ψ_2 except where combined with other quantities to form a nondimensional group. Thus,

Eq. (3.2) $$y/gt^2 = \psi_2(v_0/gt)$$

Table 3.2. Dimensional Equations for Projectile Problem

Quantity	Dimensions
Vertical displacement, y	$[L]$
Acceleration due to gravity, g	$[LT^{-2}]$
Time, t	$[T]$
Initial velocity, v_0	$[LT^{-1}]$

Exponents a and b required to make yg^at^b nondimensional may be found by writing simultaneous equations as follows (although these exponents may usually be written by inspection):

$$yg^at^b = \underset{\text{I}}{[L(LT^{-2})^aT^b]} = \underset{\text{II}}{[L^0T^0]}$$

Equating exponents of L in I and II yields:

$$1 + a = 0$$

While equating the exponents for T in I and II gives:

$$-2a + b = 0$$

When these equations are solved simultaneously, a and b are found to be -1 and -2 respectively, in agreement with the left-hand side of Eq. (3.2).

The result of a dimensional analysis is sometimes written in symbolic form in terms of nondimensional Pi quantities as follows:

Eq. (3.4) $\pi_1 = \psi(\pi_2, \pi_3, \text{etc.})$

where π_1 is a nondimensional group involving the main dependent variable (y in the projectile problem) and the other Pi values represent the remaining nondimensional quantities entering the problem.

Buckingham (1914) formulated a theorem which states that the number of Pi quantities remaining after performing a dimensional analysis is equal to the difference between the number of quantities entering the problem and the maximum number of these that are dimensionally independent. The maximum number of dimensionally independent quantities will always be equal to or less than the number of fundamental dimensions needed to write all dimensional equations. In the projectile problem, two fundamental dimensions (L, T) are involved and the maximum number of dimensionally independent quantities should, therefore, be two (g and t).

Applying Buckingham's Pi Theorem to this case:

Total quant. - dimensionally independent quant. = Pi quant.

i.e., $4 - 2 = 2$

The dimensionally independent quantities can usually be chosen in more than one way. There will often be several correct answers to a dimensional analysis. One of these answers may prove to be more convenient for a given purpose than the others. In the projectile problem, all of the pairs of dimensionally independent quantities listed in Table 3.3 could have been used and the dimensionless equations listed opposite each pair of dimensionally independent quantities would have been obtained.

Table 3.3. Possible Dimensional Analyses for Projectile Problem

Dimensionally Independent Quantities	Resulting Equation
g,t	$y/gt^2 = \psi[v_0/gt]$
g,v_0	$(yg)/v_0^2 = \psi[(gt)/v_0]$
t,v_0	$y/v_0t = \psi[(gt)/v_0]$

While there are no hard and fast rules concerning choice of dimensionally independent quantities, the following considerations serve as a useful guide:

- The main dependent variable should not be chosen
- Variables having the greatest significance should be chosen
- The variables chosen should represent as many physically different aspects of the problem as possible

In performing a dimensional analysis, it is important to include all quantities of importance to the problem. Otherwise, an incorrect and misleading result will be obtained. It is less important to include a variable about which doubt exists than to omit one which proves to be significant. Frequently, combinations of variables that are known to appear in a given class of problems in a unique association can be treated as a single quantity. This will result in fewer Pi quantities in the final result. Another method of reducing the resulting Pi quantities to a minimum is to use the maximum number of dimensionally independent quantities possible. Results of auxiliary or approximate analysis can sometimes be combined with conventional dimensional reasoning to greatly increase the power of dimensional analysis. Such auxiliary analysis includes:

- Arguments involving symmetry
- Assumptions of linearity or known behavior
- Special solutions for large or small values of variables

The meaning of some of these abstract statements will become clear after examples considered later have been considered.

5.0 CHANGE OF UNITS

Dimensional equations are useful where it is desired to change from one system of units to another. Let the dimensional equation for a given quantity (Q) be

Eq. (3.5) $$Q = [\alpha^a \beta^b]$$

If the new unit for fundamental dimension (α) is smaller than the old unit by a factor (f) while the new unit for (β) is smaller by a factor (g), then the number of units of Q in the new system of measurement per unit of Q in the old system will be ($f^a g^b$).

As an example, consider the conversion of an acceleration (a) from 32.2 ft/sec^2 into the equivalent number of in/hr^2. The appropriate dimensional equation is:

Eq. (3.6) $a = [LT^{-2}]$

The new unit for L is smaller than the old one by a factor 12 while the new unit for T is larger by a factor of 3,600. Thus, each ft/sec^2 will correspond to (12)(1/3,600)$^{-2}$ in/hr^2 and the answer to the problem is (32.2)(12)(3,600)^2 = 50 × 10^8 in/hr^2.

6.0 GALILEO REGARDING MOTION OF A PROJECTILE

In dialogs of the Fourth Day, Galileo discusses the motion of projectiles in considerable detail. It is suggested that the reading of Galileo's discourse concerning projectiles be postponed until after reading the following discussion, since the material in Galileo is scattered and difficult to follow at some points.

Galileo first proves that a projectile traveling in air will follow the path of a parabola. This is based on the properties of a parabola as a conic section given by Appolonius (~200 B.C.). After this, it is verified that assumptions that air drag and variations of gravitational attraction due to changes in distance from the center of the earth are negligible. However, it is acknowledged that a weight falling from a great height will assume a terminal velocity due to drag, and that the distance traveled to achieve terminal velocity will depend upon the shape of the body and vary inversely with its density.

In passage 281-283 of the Galileo text, an unusual approach to the velocity of a projectile is taken in which the missile is released from rest and allowed to fall freely to the ground (having the velocity attained part way down added as a uniform velocity in the horizontal direction). The point at which the horizontal velocity is introduced corresponds to the highest point reached by a projectile. Galileo refers to the elevation of the initial point of release as the sublimity, the elevation at the point where uniform horizontal velocity is introduced as the *altitude*, and twice the horizontal distance traveled before the body strikes the ground as the *amplitude*.

Since accurate means of measuring distance, time, and velocity were not available, a scheme was adopted where distances on a geometrical diagram are proportional to all three of these quantities. This procedure is described in passage 281 of the Galileo text in terms of Fig. 110.

The reason for inverting the problem, starting with free fall from the *serenity* followed by the addition of a uniform horizontal velocity at the *altitude* is that this is consistent with the very clever experiment Galileo devised to record the paths of projectiles. The experimental technique is illustrated in Fig. 3.1. This involves two plane surfaces having maximum slopes at right angles to each other. A smooth sphere is released at (*a*) and rolls down the first plane to (*b*). At this point, it has a velocity corresponding to free fall through the vertical distance between (*a*) and (*b*). At (*b*), the ball enters the second plane and moves down this with a constant horizontal velocity corresponding to that attained at (*b*), plus a vertical component of velocity corresponding to a vertical free fall from (*b*) to (*c*). The resultant velocity at (*c*) corresponds to the launching velocity at angle θ.

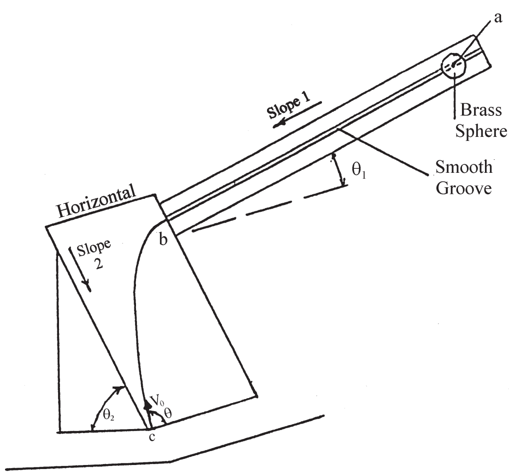

Figure 3.1. Arrangement used by Galileo in recording the path of a body subjected to uniform velocity in one direction and uniform acceleration in the orthogonal direction. This corresponds to the path taken by a projectile leaving the ground with a velocity (v_0) at an angle (θ) to the horizontal.

The method of recording the path taken from (b) to (c) is not given in the fourth day dialog, but is in passage 185 of the Galileo text. Here, it is mentioned that a parabola will be traced by a brass ball about the size of a walnut rolling down an inclined plane if the second plane in Fig. 3.1 is a metallic mirror. This is, presumably, a front surface mirror from which the silver is removed as the ball rolls over the surface. Use of a soot covered plane surface would serve equally well to record the parabolic path of the sphere moving horizontally with uniform speed, but vertically with gravitational acceleration.

The behavior of the sphere on plane 1 is described in passages 212 and 213 of the Galileo text. Here Salviati also describes how the mean time of descent was measured for different values of angle θ and distances of fall along the incline, and by repeating identical experiments many times.

It is suggested that passages 185, 212, and 213 now be read following passages 267-293 of the Galileo text.

7.0 SIMPLE PENDULUM

A pendulum consisting of a weight (W) suspended by a cord of length (ℓ) is shown in Fig. 3.2. An expression for the period of oscillation (P) is desired when friction at 0 and air resistance to motion are negligible. Consideration of this problem reveals the quantities listed in Table 3.4 to be of possible importance.

Table 3.4. Dimensions Considered in Pendulum Problem

Quantity	Symbol	Dimensions
Period	P	$[T]$
Length of cord	ℓ	$[L]$
Gravity	g	$[LT^{-2}]$
Weight of bob	W	$[F]$

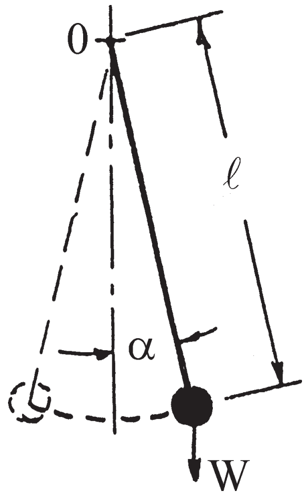

Figure 3.2. Simple pendulum.

The quantities(ℓ), (g), and (W) are dimensionally independent, hence there will be one π expression in this problem (by Buckingham's Pi Theorem: 4 - 3 = 1). This single Pi quantity will be found to be $P/(\ell/g)^{0.5}$.

The general dimensionless result expressed symbolically in Eq. (3.4) assumes a special form in this case since π_2, π_3, etc., are all unity. In this instance:

Eq. (3.7) π_1 = a nondimensional constant

and the answer to this problem is, therefore:

Eq. (3.8) $P = C\,(\ell/g)^{0.5}$

where C is the constant that may be found to be 2π by a single experiment or by the conventional application of Newton's second law.

It should be noted that the weight of the bob which was incorrectly assumed to be of importance, disappeared during the course of the dimensional analysis. This is not always the case when superfluous variables are included in the analysis. For example, if we had considered the maximum angle of swing (α) to be significant, then the end result would have been:

Eq. (3.9) $P/(\ell/g)^{0.5} = \psi(\alpha)$

This is as far as we may proceed by dimensional analysis alone. Experiments will reveal that $\psi(\alpha)$ is a constant (2π).

PROBLEMS

3.1 Explain why angle is a nondimensional quantity regardless of size or method of measurement (radians or degrees).

3.2 Combine the following variables into a nondimensional quantity or show they are dimensionally independent:

a) Force (F), mass density ($\rho = [FL^{-4}T^2]$), velocity (V), and a diameter (D)

b) Fluid flow rate ($Q = [FT^{-1}]$), a length ℓ, and viscosity ($\mu = [FTL^{-2}]$)

c) Surface tension ($T_e = [FL^{-1}]$, viscosity (μ), and mass density (ρ)

3.3 Are the following quantities dimensionally independent? If not, form them into one or more nondimensional groups.

a) Acceleration (a), surface tension ($T_e = [FL^{-1}]$), and area (A)

b) A volume (B), an angle (α), a velocity (V), and acceleration (a)

c) Mass density ($\rho = [ML^{-3}]$, viscosity ($\mu = [FTL^{-2}]$), and bulk modulus ($K = [FL^{-2}]$)

d) Acceleration (a), surface tension ($T_e = [FL^{-1}]$), velocity (V), and mass ($M = [FL^{-1}T^2]$)

3.4 Are the following quantities dimensionally independent? If not, form them into one or more nondimensional groups.

a) Mass density ($\rho = [ML^{-3}]$), velocity (V), diameter (D), and surface tension (T_e)

b) Mass density ($\rho = [ML^{-3}]$), surface tension ($T_e = [FL^{-1}]$), and viscosity ($\mu = [FTL^{-2}]$)

c) Mach number ($N_M = [F^0, L^0, T^0]$, kinematic viscosity ($\nu = [L^2T^{-1}]$), and surface tension ($T_e = [FL^{-1}]$)

3.5 When a capillary tube of small inside diameter is immersed in a liquid of surface tension $T_e = [FL^{-1}]$ and specific weight (γ), the liquid will rise to a height (h) as shown in Fig. P3.5 and $h = \Psi(T_e, d, \gamma)$. Perform a dimensional analysis and, thus, reduce the number of variables.

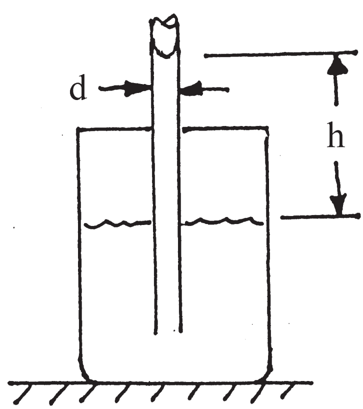

Figure P3.5.

3.6 It is reasonable to assume that (h) will be proportional to (γ). What result is obtained when this is incorporated into the solution of Problem 3.5?

3.7 The drag force (F_D) on a body of a given shape (such as a sphere) will be influenced by the variables listed below having dimensions indicated.

Quantity	Dimensions
Drag force, F_D	$[F]$
Size of body, S	$[L]$
Density of fluid, ρ	$[FL^{-4}T^{2}]$
Relative velocity, V	$[LT^{-1}]$
Viscosity of fluid, μ	$[FTL^{-2}]$
Acceleration due to gravity, g	$[LT^{-2}]$
Velocity of sound in fluid, C	$[LT^{-1}]$

The drag force is the main dependent variable in this case, and there will be three dimensionally independent quantities involving F, L, and T. These may be taken to be S, ρ, and V.

a) Show that S, ρ, and V are dimensionally independent.

b) How many nondimensional quantities ($\pi_1, \pi_2, \pi_3 ...$) will there be, in this case?

c) Perform a dimensional analysis and express π_1 involving F_D as a function of the other π quantities; i.e., find $\pi_1 = \psi(\pi_2, \ldots)$.

3.8 The velocity of a wave in deep sea water (V) is a function of the wavelength (λ) and the acceleration due to gravity (g) (Fig. P3.8). By dimensional analysis, derive an expression for the wave velocity in deep water. Will the wave length be smaller in a high wind?

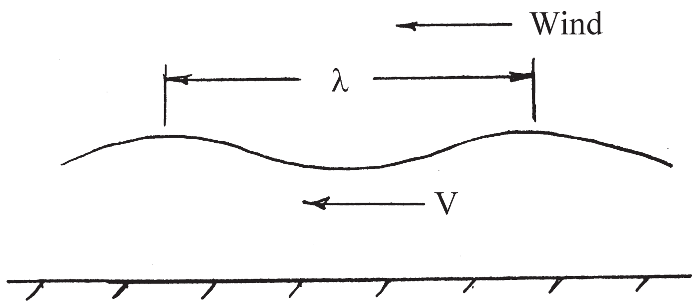

Figure P3.8.

3.9 Water has a surface tension of 73 dynes/cm. To how many (in.lb/in^2) does this correspond?

3.10 A small spherical drop of liquid (diameter d) such as one of those found on a cabbage leaf by Galileo (passage 115) oscillates between a sphere and an ellipse under the action of surface tension when the leaf is disturbed (Fig. P3.10). If the drop is very small, gravitational forces will be negligible and the period of the oscillation (P) will be:

$$P = \psi(T_e, d, \rho)$$

where ρ is the density of the liquid ($FL^{-4}T^2$). Perform a dimensional analysis for this problem. Will the frequency of oscillation of the drop when disturbed increase or decrease as the drop gets smaller?

Figure P3.10.

3.11 Surface tension (T_e) is sometimes determined by noting the volume (B) of a drop delivered from a narrow dropping tube of diameter (d) at its tip (Fig. P3.11) (Tate's pendant drop method). The formation of the pendant drop and its subsequent detachment are complex phenomena but the following variables are of importance:

$$B = \psi(d, \gamma, T_e)$$

where γ is the specific weight of the fluid. Perform a dimensional analysis for this problem with T_e the main dependent variable.

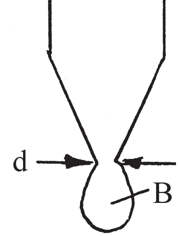

Figure P3.11.

3.12 If the velocity of water (V) flowing through the orifice in the tank (Fig. P3.12) is a function of gravity (g) and the height of the fluid (h):

$$V = \psi(g, h)$$

a) Perform a dimensional analysis.

b) By what factor will the velocity change when (h) is doubled?

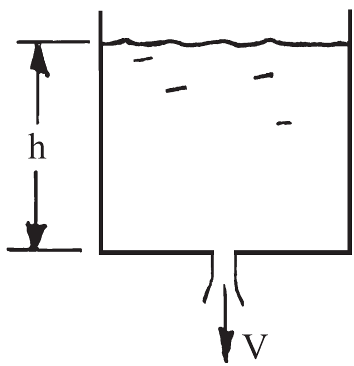

Figure P3.12.

3.13 Repeat Problem 3.7, taking M, L, and T as the fundamental set of dimensions instead of F, L, and T.

3.14 Repeat Problem 3.12 taking M, L, and T as the fundamental set of dimensions instead of F, L, and T.